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Aggregation–disorder transition induced by fluctuating random forces

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Abstract. We investigated the motion of many non-interacting particles with inertia in a viscous medium subject to an external, random force field that varied both in space and time. By varying parameters in the problem, it was found that there is a transition between two regimes. One regime, characterised by low mass or high viscosity, leads to the aggregation of particles. The other regime, high mass or low friction, leads to a complete lack of aggregation. The transition between these two regimes is marked by a discontinuous change in the order parameters defined for this problem. The relation between this work and previous work on iterated functions is discussed.

1. Introduction

The problem of particles in a random environment has recently received much attention (Marinari *et al* 1983, Derrida and Pomeau 1982). The equation studied has been of the form

$$\dot{x} = f(x) + \zeta(t) \tag{1}$$

with $f(x)$ and $\zeta(t)$ random functions with zero mean and short-range correlations. It was recently shown that in one dimension x has extremely slow motion and that the power spectrum exhibits $1/f$ noise (Marinari *et al* 1983).

A related problem which has received less attention is that of particles in a random environment which is a function of position and time. Here we investigate this problem and find some surprising results. The equation we are considering is

$$m\ddot{x} + \nu\dot{x} = f(x, t) \tag{2}$$

where $f(x, t)$ is a random force with a mean of zero and

$$\langle f(x', t')f(x, t)\rangle = g(x' - x)\delta(t' - t) \tag{3}$$

where $g(x)$ is a rapidly decaying function which has a parabolic maximum at $x = 0$. This represents a particle with inertia in a viscous medium under the influence of a randomly fluctuating force that is a function of position and time. A quantum mechanical extension of this equation (with $\nu = 0$ but on a lattice) has been investigated (Ovchinnikov and Erikhman 1975, Marianer *et al* 1982, Madhukar and Post 1977) and the results used to explain the conductivity of certain organic conductors.

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2. Preliminary remarks

If we consider one particle by itself, the solution to (2) is well known. The position dependent force $f(x, t)$ can be replaced by a force that is only time dependent $f(t)$. For short times $\langle x^2 \rangle \propto t^2$ and for long times $\langle x^2 \rangle \propto t$. But now consider many particles initially equally spaced in one dimension all subject to the force $f(x, t)$ and obeying (2). Then we find the following qualitative results. When $m = 0$ or below some critical value that depends on the friction coefficient ν and $g(x)$, particles aggregate. That is, as t increases, particles come together forming aggregates. These aggregates in turn come together forming still larger aggregates. It must be stressed that this occurs in the absence of any interaction between the particles. Furthermore, if we denote the distance between two particles that are initially very close together by $\Delta x(t)$ then

$$\langle (\Delta x(t))^2 \rangle \propto e^{ct} \tag{4}$$

where c is a positive constant. At first sight this might seem to contradict the notion that the particles aggregate, but in fact the two results are completely consistent as we shall later see. When m is above the critical value mentioned above the particles do not aggregate but diverge away from each other so that the initial order of the particles is lost.

3. Mathematical analysis

In order to investigate this problem analytically, we shall use the general method discussed by Klyatskin and Tatarskii (1974) for finding the differential equation satisfied by the probability distribution of stochastic variables. In their paper Klyatskin and Tatarskii discuss many different situations. One of these is the one considered in this paper with $m = 0$. Klyatskin and Tatarskii derived the differential equation for the probability distribution of the relative distance between two particles as a function of time, $P(\Delta x, t)$ satisfied by equation (2) (with $m = 0$). They find

$$\frac{\partial P(\Delta x, t)}{\partial t} = \frac{\partial^2 (D(\Delta x) P(\Delta x, t))}{\partial \Delta x^2} \tag{5}$$

where $D(x) = 2(g(0) - g(x))$. When $\Delta x(t)$ is much less than the correlation length of $g(x)$, we can replace $D(x)$ by kx^2 where the constant $k = -2g''(0)$. Klyatskin and Tatarskii use this equation to calculate only the second moment of the probability distribution which leads them to (4). However, this result does not contain the essential physics of the problem. This can be seen by solving (5) which leads to

$$P(\Delta x, t) = \frac{\exp\{-[\ln(\Delta x / x_0) + 3t]^2 / 4t + 2t\}}{x_0 (4\pi t)^{1/2}}. \tag{6}$$

Now we define the following order parameter

$$\theta = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \int_{-\varepsilon}^{\varepsilon} P(x, t)\, dx. \tag{7}$$

This represents the probability that two particles will be found an infinitesimally small distance away from each other in the infinite time limit. By substituting (6) into (7) it can easily be seen that $\theta = 1$, which implies that neighbouring particles aggregate. So

the complete probability distribution for particle separation given by (6) implies aggregation whilst also giving the exponential divergence in (4). This is because the major contribution to the average in (4) does not come from the particles that aggregate, but from those particles between two aggregates. The number of such particles is of measure zero in the infinite time limit, but their relative separation diverges exponentially with time. It is these particles that dominate the average and lead to (4).

Now we discuss the relative separation of two particles for finite m. Applying the same method used by Klyatskin and Tatarskii one can easily obtain the probability distributions of the relative positions and velocities as a function of time, $P(x, v, t)$ which is

$$\frac{\partial P}{\partial t}+\frac{\partial vP}{\partial x}-\frac{\nu}{m}\frac{\partial vP}{\partial v}-\frac{k}{m^2}x^2\frac{\partial^2 P}{\partial v^2}=0. \tag{8}$$

This contains three parameters, m, ν and k so that if we make the rescaling $t \leftarrow tk^{1/3}m^{-2/3}$ and $v \leftarrow vk^{-1/3}m^{2/3}$ then

$$\frac{\partial P}{\partial t}+\frac{\partial vP}{\partial x}-a\frac{\partial vP}{\partial v}-x^2\frac{\partial^2 P}{\partial v^2}=0 \tag{9}$$

where $a=\nu(km)^{-1/3}$. Now we consider an initial probability distribution at time $t=0$ which is symmetric around $x=0$, i.e. $P(x, v, t=0)=P(-x, -v, t=0)$. This choice does not alter the asymptotic behaviour of the particles but is convenient as we only need consider $P(x, v, t)$ for $x>0$. Then we make the substitutions $u=v/x$ and $y=\ln(x)$ which gives the equation for $P(y, u, t)$

$$\frac{\partial P}{\partial t}-u^2\frac{\partial P}{\partial u}+u\frac{\partial P}{\partial y}-a\frac{\partial uP}{\partial u}-\frac{\partial^2 P}{\partial u^2}=0. \tag{10}$$

Now we define the two-sided Laplace transform with respect to u and y of $P(y, u, t)$:

$$p(k, \lambda, t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ky}\, e^{\lambda u}\, P(y, u, t)\, dy\, du \tag{11}$$

which gives

$$\frac{\partial p}{\partial t}+(a\lambda-k)\frac{\partial p}{\partial \lambda}+\frac{\partial^2 \lambda p}{\partial \lambda^2}-\lambda^2 p=0. \tag{12}$$

Now consider the equation when $\lambda=0$ which gives

$$\left.\frac{\partial p}{\partial t}\right|_{\lambda=0}=(k-2)\left.\frac{\partial p}{\partial \lambda}\right|_{\lambda=0}. \tag{13}$$

We see that an eigenfunction of (12) will have the form $\exp(-\gamma(k, a)t)f(\lambda; k, a)$, since k and a are parameters in (12). We will now make the assumption that the long-time behaviour of p will be dominated by the smallest eigenvalue $\gamma(k, a)$. Substituting this form of p into (13), (which should be correct for large times) we obtain

$$\gamma(k, a)=(2-k)\left.\frac{\partial \ln(f(\lambda; k, a))}{\partial \lambda}\right|_{\lambda=0}. \tag{14}$$

It is now possible to use (14) to find the criterion for the transition described above. We can easily generalise the order parameter θ defined by (7) to include the velocity

dependent probability distribution $P(x, v, t)$

$$\theta = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{\infty} P(x, v, t)\, dv\, dx. \tag{15}$$

Using the symmetry of the initial conditions we can rewrite (7) as

$$\begin{aligned} 1 - \theta &= 2 \lim_{\varepsilon \to 0} \lim_{t \to \infty} \int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} P(x, v, t)\, dv\, dx \\ &= 2 \lim_{s \to -\infty} \lim_{t \to \infty} \int_{s}^{\infty} \int_{-\infty}^{\infty} P(y, u, t)\, e^{2y}\, du\, dy \\ &\leqslant 2 \lim_{s \to -\infty} \lim_{t \to \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(y-s)\delta} P(y, u, t)\, e^{2y}\, du\, dy \\ &= 2 \lim_{s \to -\infty} e^{-s\delta} \lim_{t \to \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{y(2+\delta)} P(y, u, t)\, du\, dy \\ &= 2 \lim_{s \to -\infty} e^{-s\delta} \lim_{t \to \infty} p(k = 2 + \delta, \lambda = 0, t) \end{aligned} \tag{16}$$

where δ can be any constant greater than 0. So if $p(2+\delta, 0, t)$ goes to 0 as t tends to infinity, then (16) implies that $\theta = 1$. Using an analogous argument it is easy to show that an order parameter representing 'disorder'

$$D = 2 \lim_{z \to \infty} \lim_{t \to \infty} \int_{z}^{\infty} \int_{-\infty}^{\infty} P(x, v, t)\, dv\, dx \tag{17}$$

is 1 if $p(2-\delta, 0, t)$ goes to 0 as t goes to infinity.

It is easily seen from (14) that if

$$\left. \frac{\ln(f(\lambda; k, a))}{\partial \lambda} \right|_{\lambda = 0} \tag{18}$$

is positive at and close to $k = 2$ then $\gamma(k, a) < 0$ for $k > 2$ and $\gamma(k, a) > 0$ for $k < 2$. This would imply that the order parameter $\theta = 1$ and $D = 0$. Similarly if the quantity in (18) is negative then this would imply that $\theta = 0$ and $D = 1$. Thus the transition between aggregation and disorder, which is a function of the parameter a, should occur when the quantity in (18) is zero at $k = 2$. Thus the transition between aggregation and disorder is determined by the sign of (18) evaluated at $k = 2$. Equation (18) evaluated at $k = 2$ can be easily expressed in terms of the physical variables v and x and is

$$\int_{0}^{\infty} \int_{-\infty}^{\infty} v \ln(x) \frac{\partial f(v; x, a)}{\partial t}\, dv\, dx. \tag{19}$$

4. Results and discussion

In the above analysis we have assumed that for a given value of a (not at the transition) there exists some sufficiently small neighbourhood around $k = 2$ where (18) is always positive or always negative. In other words, we are assuming that (19) is not zero. Of course at the critical value of a, this may not be the case since (19) is zero at $k = 2$.

There exists, however, the possibility that the above assumption is wrong and that (19) is always zero at $k = 2$.

Because of the assumptions made in the above analysis, a numerical simulation of (2) has been performed to check that the aggregation transition actually occurs. The force $f(x, t)$ was produced by selecting random numbers from a uniform distribution between $-A$ and A. Each random number is associated with a different point on an evenly spaced two-dimensional grid (x_i, t_j). These random numbers define the values of the force $f(x_i, t_j)$. To obtain the value of $f(x, t)$ for arbitrary x and t, linear interpolation between grid points nearest x and t was performed. To obtain the behaviour of many particles subject to the same random potential, (2) was solved numerically for different initial values of x but using the same $f(x, t)$. Figure 1 shows some typical results. The horizontal direction indicates time on some arbitrary scale. The vertical axis indicates the position of the particles as a function of time. At the start of the simulation the particles were equally spaced and given zero initial velocity. The above simulation appears to be in agreement with the theoretical predictions. It is clear that in figure 1(a), when $m = 0.5$, $\nu = 1$ and $A = 1$ that the particles aggregate with time. However, in figure 1(b), when $m = 1$, $\nu = 0.15$ and $A = 1$ the particle positions become disordered. Figure 2 plots the data in figure 1(a) in a different way. The vertical axis of each frame represents the positions of the particles as a function of their initial positions. Each frame plots the positions at different times. From analysing data as a function of the parameters, it appears that the transition occurs approximately at $a = 0.4$.

It is of interest to point out the close link between this problem and that of iterating random functions described in a previous paper (Deutsch 1984). I previously conjectured that there was a connection and the above results confirm this. I investigated the function $g_n(x)$ defined as

$$g_n(x) = f_n(f_{n-1}(\ldots f_2(f_1(x))\ldots)) \tag{20}$$

where each $f_i(x)$ has the form

$$f_i(x) = x + \varepsilon_i(x). \tag{21}$$

Here $\varepsilon_i(x)$ is a random function with zero mean and $\varepsilon_i(x)$ and $\varepsilon_j(x)$ are statistically independent when $i \neq j$. It was found that when the $f_i(x)$ were monotonic, for large n, $g_n(x)$ looked like a series of sharp steps with the average height and width of each step proportional to $n^{1/2}$. This result can be interpreted as the aggregation of particles in analogy with figure 2. The horizontal axis labels the initial positions of the particles and the vertical axis their final positions after n steps. Each iteration $f_i(x)$ describes the evolution of the particles position to the next time step.

The crucial parameter controlling the aggregation of these particles under the dynamics of (1) was found to be the probability distribution for the derivatives of $g_n(x)$ which I label $p_n(a)$, i.e. the probability that the $(\mathrm{d}g_n(x)/\mathrm{d}x)$ equals a. The point x where the derivative is evaluated is irrelevant since the function $\varepsilon_i(x)$ has statistical properties that are translationally invariant. In terms of the particle picture, $p_n(a)$ has a very simple meaning. If two particles start with a very small separation Δx, then their final position will be $(\Delta x\, \mathrm{d}g_n(x)/\mathrm{d}x)$. Therefore $p_n(a)$ is the probability that two particles will have a separation $a\Delta x$ after n steps. When $m = 0$ in (2) $P(\Delta x, t)$ given by (6) has the identical functional form to $p_n(a)$ (for n large) when the $f_i(x)$ are monotonic.

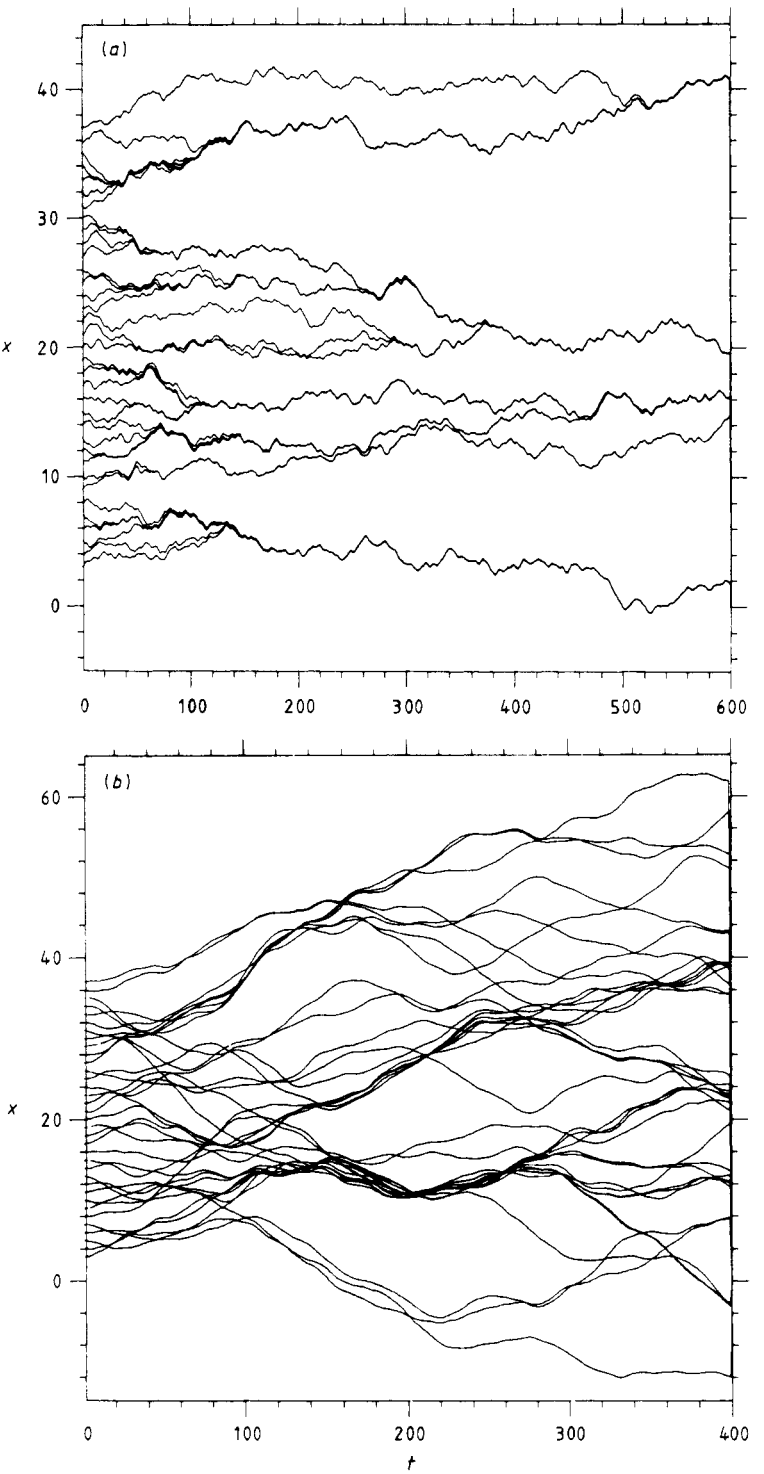

Figure 1. The position of 37 particles in a random time varying potential as a function of time. (*a*) The value of the parameters $m = 0.5$, $A = 1$ and $\nu = 1$, is such that the particles aggregate. (*b*) The value of the parameters $m = 1$, $A = 1$, $\nu = 0.15$ is such that the particles do not aggregate.

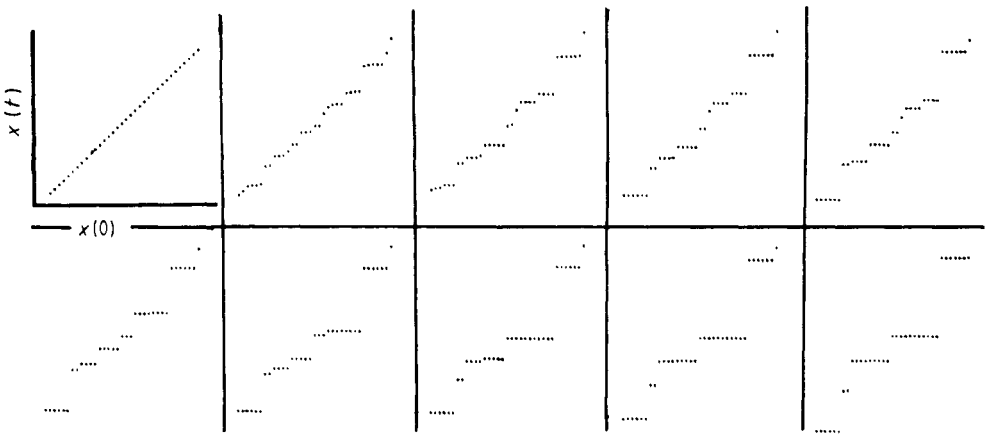

Figure 2. The position of 37 particles as a function of their initial positions for the same parameters as in figure 1(*a*). Every frame is in time steps of 30.

When the $f_i(x)$ are not monotonic, it was rigorously shown that there is a transition from aggregation ($\theta = 1$, $D = 0$) to disorder ($\theta = 0$, $D = 1$), that occurs as the degree of non-monoticity (i.e. the width of $p_n(a)$) is varied. Here we find a strikingly similar transition when we vary m, ν and k. Thus the work in this paper has supported the conjecture that the long-time behavior of these two problems are identical.

There are other problems where there exists a connection between iterated maps and solutions to differential equations. For example, if we consider (2) in the case where $f(x, t)$ is not random but equal to $a \sin t - bx^3 + cx$ (such as the nonlinear Duffing oscillator), then by varying the friction coefficient ν, a sequence of period doublings is observed which ultimately give rise to chaos beyond a critical value of ν. This equation has been shown numerically (Huberman and Crutchfield 1979) to be in the same universality class as the iterated map $f(x) = rx(1 - x)$. The connection between (2) and (19) in this paper is somewhat different for two reasons. First we are interested in the behaviour of a large number of particles simultaneously, as one particle by itself shows no interesting features. In the case of the logistic map and the Duffing oscillator, interesting features are observed by considering the trajectory of one point. Second, the equations of motion in this paper are stochastic. In the Duffing oscillator, chaotic behaviour is observed but the equations of motion are deterministic.

I am currently investigating the properties of (2) in higher dimensions.

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